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Scattering of surface waves by rectangular obstacles in waters of finite depth

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The scattering of infinitesimal surface waves normally incident on a rectangular obstacle in a channel of finite depth is considered. A variational formulation is used as the basis of numerical computations. Scattering properties for bottom and surface obstacles of various proportions, including thin barriers and surface docks, are presented. Comparison with experimental and theoretical results by other investigators is also made.

1. Introduction

The scattering of water waves by large obstacles has been well known for the mathematical difficulties encountered within the framework of linearized potential theory. While some general features (such as symmetry relations, bounds) regarding the reflexion or transmission properties have been studied previously (Kreisel 1949), explicit calculations have been successful only in a few limiting cases. Most of the known exact solutions are in waters of infinite depth for vertical thin barriers (Dean 1945; Ursell 1947), semi-immersed circular cylinders (Dean & Ursell 1959), or a step shelf (Newman 1965*a*). Furthermore, the only exact solution for a continuously varying depth is that of Roseau (1952) for a special bottom profile. Barakat (1968) has recently given numerical solutions for symmetrical cylinders with rounded corners fixed in the free surface.

In the case of finite water depth some approximate solutions have been found for long waves (Kajiura 1961; Ogilvie 1960), for long bottom obstacles (Newman 1965*b*), and for low bottom obstacles (Kreisel 1949; Mei 1967). Stoker (1957, p. 434) dealt with the problem of a surface dock with zero submergence in long waves. While these studies point out many interesting qualitative features of the wave scattering, their approximations also severely limit the quantitative usefulness of the results.

For engineering purposes it is of interest to obtain results valid for the whole wavelength spectrum, finite water depth, and finite obstacle dimensions. The effect of various geometrical proportions can best be understood by investigating rectangular obstacles in a channel of finite depth. This class of problems has close analogues in electromagnetic wave guides with a thick inductive window, for which a variational method of solution is most effective.† Miles (1967) recently

† Theoretical attempts by straightforwardly joining eigenfunction expansions at junction surfaces have been unsuccessful, because of poor convergence in numerical work (Takano 1960).

applied the variational method to a step shelf and achieved remarkably good results, as compared with the numerical solution by Newman (1965*a*), for the far field with only a crude approximation on the near field. In this paper the same formulation is adopted and numerical results for both surface (semi-immersed) and bottom obstacles of various proportions (figures 1(*a*), (*b*)) are presented. Only the normal incidence of simple harmonic waves on two-dimensional obstacles is treated here. Available experimental, as well as theoretical, data are collected for comparison.

2. General formulation

In this paper explicit calculations will be done for an obstacle fixed either on the bottom (case I) or in the free surface (case II), see figure 1, although the combined problem of a thick barrier having a submerged gap can be similarly treated. With the usual assumptions of a perfect fluid and small amplitudes the velocity potential,

$$\Phi(x,y,t) = \phi(x,y)\,e^{i\omega t}, \tag{2.1}$$

satisfies the following conditions:

$$\nabla^2\phi = 0, \quad x, y \text{ in fluid}; \tag{2.2}$$

$$\frac{\partial\phi}{\partial y} - \sigma\phi = 0, \quad \sigma \equiv \omega^2/g, \quad y = 0, \quad \begin{Bmatrix} |x| < \infty: \text{(I)}, \\ |x| > l: \ \text{(II)} \end{Bmatrix}; \tag{2.3}$$

$$\frac{\partial\phi}{\partial y} = 0, \quad y = -h, \quad \begin{Bmatrix} |x| > l: \ \text{(I)}, \\ |x| < \infty: \text{(II)} \end{Bmatrix}; \quad y = -H, \quad |x| < l: \text{(I) and (II)}; \tag{2.4}$$

$$\frac{\partial\phi}{\partial x} = 0, \quad |x| = l, \quad \begin{Bmatrix} -h < y < -H: \text{(I)}, \\ -H < y < 0: \ \ \text{(II)} \end{Bmatrix}. \tag{2.5}$$

In addition we impose the radiation condition that there are both left- and right-going waves at $x \sim -\infty$ but only right-going waves at $x \sim \infty$.

As is usual in problems of wave-guide discontinuities it is convenient to split the potential into a symmetric and an antisymmetric part

$$\phi = \phi_S + \phi_A \quad \text{with} \quad \phi_S(-x,y) = \phi_S(x,y), \quad \phi_A(-x,y) = -\phi_A(x,y) \tag{2.6}$$

such that the analysis can be restricted to $x < 0$ only. Physically $\phi_S(\phi_A)$ corresponds to the scattering of two waves incident from $x \sim -\infty$ and $x \sim \infty$ towards a symmetrical obstacle, with equal amplitude and equal (opposite) phase. The symmetrical case has also been studied by Miles (1967). Since the energy flux across the plane $x = 0$ is

$$E_{(\,)} = -\int\left[p_{(\,)}\frac{\partial\Phi_{(\,)}}{\partial x}\right]_{x=0} dy = \rho\int\left[\frac{\partial\Phi_{(\,)}}{\partial t}\frac{\partial\Phi_{(\,)}}{\partial x}\right]_{x=0} dy, \quad \Phi_{(\,)} = \Phi_S \text{ or } \Phi_A$$

and since
$$\frac{\partial\Phi_S}{\partial x}(0,y,t) = \frac{\partial\Phi_A}{\partial t}(0,y,t) = 0,$$

it follows that
$$E_S = E_A = 0 \quad \text{at} \quad x = 0.$$

Thus the reflexion coefficient is of unit magnitude in either case;

$$|R_S| = |R_A| = 1. \tag{2.7}$$

However, it is the phase differences between the complex reflexion coefficients R_S and R_A that contribute to the non-trivial character of the resultant scattering. Specifically, the total reflexion and transmission coefficients are

$$\left.\begin{aligned} R &\equiv |R|\,e^{i\theta_R} = \tfrac{1}{2}(R_S+R_A),\\ T &\equiv |T|\,e^{i\theta_T} = \tfrac{1}{2}(R_S-R_A).\end{aligned}\right\} \tag{2.8}$$

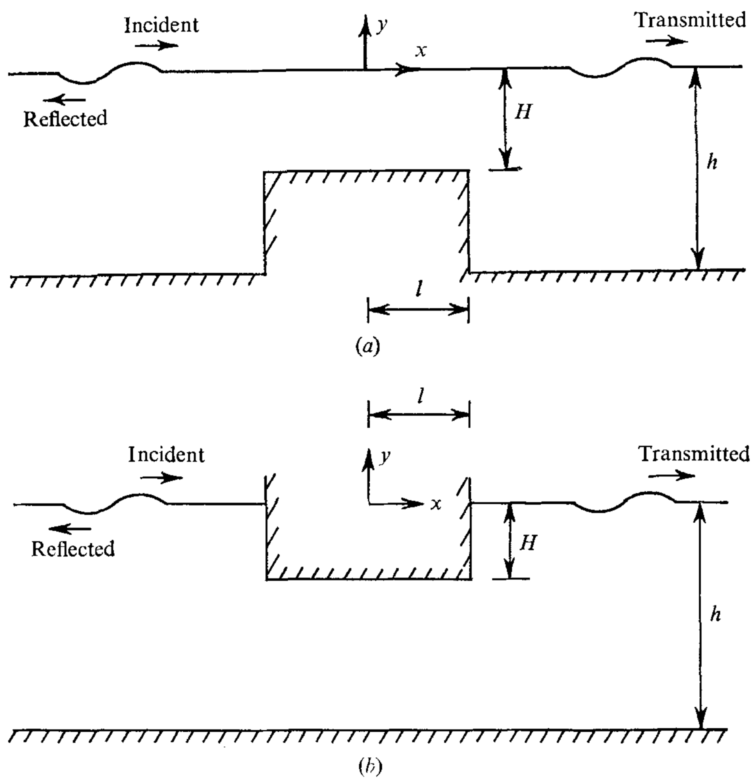

FIGURE 1. Definition sketch. (*a*) Submerged obstacle. (*b*) Surface obstacle.

When the complex coefficients R_S and R_A are represented by unit vectors in a phasor diagram, one can deduce the following known results (Kreisel 1949):

$$\left.\begin{aligned} |R|^2+|T|^2 &= 1,\\ \mathbf{R}.\mathbf{T} &= 0.\end{aligned}\right\} \tag{2.9}$$

The phase angles θ_R and θ_T differ by $\tfrac{1}{2}\pi$; however, their respective values depend on the quadrants in which the vectors $\mathbf{R}_S$ and $\mathbf{R}_A$ lie.

3. Case I. Bottom obstacle

(*a*) *The antisymmetric part and the variational formulation*

Although the variational formulation to be employed is a standard procedure in wave-guide problems (Collin 1960) and has also been demonstrated by Miles (1967, 1968) for hydrodynamical problems, we shall outline it here for the sake of convenience.

The potential for different regions will be expressed by appropriate eigenfunction expansions; for explicitness we separate the propagating mode from the evanescent modes

$$\phi_A(x,y) = b_1[e^{-ik(x+l)} + r_A\, e^{ik(x+l)}]f_1 + \sum_{n=2}^{\infty} b_n\, e^{q_n(x+l)} f_n \quad (x < -l,\ -h < y < 0) \quad (3.1a)$$

$$= B_1 F_1 \sin Kx + \sum_{n=2}^{\infty} B_n F_n \sinh Q_n x \quad (|x| < l,\ -H < y < 0); \quad (3.1b)$$

where

$$\left.\begin{aligned} f_1(y) &= \sqrt{2}(h + \sigma^{-1}\sinh^2 kh)^{-\frac{1}{2}} \cosh k(y+h); \\ f_n(y) &= \sqrt{2}(h - \sigma^{-1}\sin^2 q_n h)^{-\frac{1}{2}} \cos q_n(y+h) \quad (n = 2, 3, \ldots); \end{aligned}\right\} \quad (3.2)$$

and k, q_n are the real positive roots of the following equations:

$$k \tanh kh = \sigma, \quad q_n \tan q_n h = -\sigma \quad (n = 2, 3, \ldots). \quad (3.3)$$

It can be readily shown that $\{f_i\}$ form an orthonormal set over the interval $(-h < y < 0)$. For the shallow region the quantities F_i, K, and Q_n are defined similarly by replacing f_i, k, and q_n above with h changed to H. We note that the reflexion coefficient is given by

$$R_A = r_A\, e^{2ikl}. \quad (3.4)$$

Continuity of ϕ_A at $x = -l$ requires that

$$b_1(1 + r_A)f_1 + \sum_2^{\infty} b_n f_n = -B_1 F_1 \sin Kl - \sum_2^{\infty} B_n F_n \sinh Q_n l. \quad (3.5)$$

Continuity of the horizontal velocity $\partial\phi_A/\partial x$ at $x = -l$ requires that

$$\left.\begin{aligned} U_A(y) &\equiv -ikb_1(1 - r_A)f_1 + \sum_2^{\infty} b_n q_n f_n, \\ &= B_1 K F_1 \cos Kl + \sum_2^{\infty} B_n Q_n F_n \cosh Q_n l, \end{aligned}\right\} \quad (-H < y < 0);$$

$$= 0 \quad (-h < y < -H). \quad (3.6)$$

When the orthonormal conditions are used the coefficients b_i, B_i can be expressed as integrals of U_A and f_i, F_i from (3.6); substitution into (3.5) yields the following integral equation for $U_A(y)$:

$$S_A f_1 \int_{-H}^{0} U'_A f'_1 \, dy' = \int_{-H}^{0} U'_A G_A(y|y')\, dy', \quad (3.7)$$

where

$$S_A \equiv (1 + r_A)(ik)^{-1}(1 - r_A)^{-1}, \quad (3.8)$$

$$U'_A \equiv U_A(y'), \text{ etc.} \quad (3.9)$$

and

$$G_A(y|y') = \sum_2^{\infty} f_n f'_n / q_n + \frac{\tan Kl}{K} F_1 F'_1 + \sum_2^{\infty} \frac{\tanh Q_n l}{Q_n} F_n F'_n. \quad (3.10)$$

Multiplying both sides of (3.7) by U_A and integrating with respect to y from $-H$ to 0, we obtain finally

$$S_A = \iint_{-H}^{0} U_A G_A(y|y')\, U'_A \, dy\, dy' \Big/ \left\{\int_{-H}^{0} U_A f_1 \, dy\right\}^2. \quad (3.11)$$

It is easy to show that, in the form (3.11), S_A is stationary with respect to independent variations of U_A or U'_A. By virtue of (3.8), S_A is obviously related to the reflexion characteristics. We shall first assume a form for U_A with many parameters $\{u_r\}$; the 'best' values for $\{u_r\}$ will then be chosen by invoking the fact that S_A is stationary.

(b) *Calculation of S_A*

We follow the steps explained in Collin (1960, p. 352) and expand U_A in terms of eigenfunctions $F_r(y)$,

$$U_A(y) = \sum_1^M u_r F_r(y), \tag{3.12}$$

with the coefficients u_r yet unknown. Substituting (3.12) into (3.11) and invoking the conditions

$$\frac{\partial S_A}{\partial u_r} = 0 \quad (r = 1, 2, 3, \ldots, M),$$

a set of homogeneous equations for u_r results. For non-trivial solutions the vanishing of the coefficient determinant $(M \times M)$ finally yields an expression for

$$S_A = \begin{vmatrix} \cdots & \left(\dfrac{g^A_{1s}}{P_{11}P_{1s}}\right) & \cdots \\ & \vdots & \\ \cdots & \left(\dfrac{g^A_{1s}}{P_{11}P_{1s}} - \dfrac{g^A_{rs}}{P_{1r}P_{1s}}\right) & \cdots \\ & \vdots & \\ \cdots & \cdot & \cdots \end{vmatrix} \div \begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 \\ & & \vdots & & \\ \cdots & & \left(\dfrac{g^A_{1s}}{P_{11}P_{1s}} - \dfrac{g^A_{rs}}{P_{1r}P_{1s}}\right) & & \cdots \\ & & \vdots & & \\ \cdots & & \cdot & & \cdots \end{vmatrix}, \tag{3.13}$$

where
$$\begin{aligned} P_{nm} \equiv \int_{-H}^{0} f_n F_m \, dy = {} & (h - \sigma^{-1}\sin^2 q_n h)^{-\frac{1}{2}} (H - \sigma^{-1}\sin^2 Q_n H)^{-\frac{1}{2}} \\ & \times \{(q_n + Q_m)^{-1}[\sin(q_n h + Q_m H) - \sin q_n(h-H)] \\ & + (q_n - Q_m)^{-1}[\sin(q_n h - Q_m H) - \sin q_n(h-H)]\}, \end{aligned} \tag{3.14}$$

and

$$g^A_{rs} = \sum_{n=2}^{\infty} P_{nr}P_{ns}/q_n + \frac{\tan Kl}{K}\delta_{1r}\delta_{1s} + \sum_{n=2}^{\infty} \frac{\tanh Q_n l}{Q_n}\delta_{nr}\delta_{ns} \quad (r, s = 1, 2, 3, \ldots, M). \tag{3.15}$$

In order not to repeat (3.14) for P_{1m} and P_{n1}, we mention that q_n should be replaced by ik for $n = 1$, and Q_m by iK for $m = 1$.

Thus the problem reduces to the numerical evaluation of the determinants where each term is composed of infinite series. Computational aspects are discussed in § 6.

(c) *The symmetric part*

Except for those indicated below, most of the relations in §§ 3 (*a*) and 3 (*b*) remain applicable with the mere change of subscripts from $(\ )_A$ to $(\ )_S$. The potential for the shallow region should be changed to

$$\phi_S = B_1 F_1 \cos Kx + \sum_2^{\infty} B_n F_n \cosh Q_n x \quad (|x| < l, \ -H < y < 0). \tag{3.16}$$

Formulas replacing (3.10) and (3.15) are

$$G_S(y/y') = \sum_2^\infty \frac{f_n f_n'}{q_n} - \frac{\cot Kl}{K} F_1 F_1' + \sum_2^\infty \frac{\coth Q_n l}{Q_n} F_n F_n', \tag{3.17}$$

and

$$g_{rs}^S = \sum_{n=2}^\infty \frac{P_{nr} P_{ns}}{q_n} - \frac{\cot Kl}{K} \delta_{1r} \delta_{1s} + \sum_{n=2}^\infty \frac{\coth Q_n l}{Q_n} \delta_{nr} \delta_{ns}. \tag{3.18}$$

4. Case II. Surface obstacles

The difference from the case of bottom obstacles arises solely from the different boundary conditions in the region $|x| < l$. Without introducing new notations, we shall only identify those formulas in § 3 that need modifications and give the corresponding replacements:

$$\phi_A = B_1 F_1 x + \sum_2^\infty B_n F_n \sinh Q_n x, \qquad \text{(4.1) for (3.1}b\text{)}$$

$$\phi_S = \sum_2^\infty B_n F_n \cosh Q_n x, \qquad \text{(4.2) for (3.16)}$$

$$\left\{\begin{matrix} |x| < l, \\ -h < y < -H \end{matrix}\right\};$$

where

$$F_1 = D^{-\frac{1}{2}}, \quad F_n = (2/D)^{\frac{1}{2}} \cos Q_n(y+h), \tag{4.3}$$

$$Q_n = (n-1)\pi/D, \quad D \equiv h - H. \tag{4.4}$$

The functions $\{F_i\}$ form the complete orthonormal set appropriate for the boundary conditions (2.4) (case II). In ϕ_S we have omitted a constant term $B_1 F_1$ which is immaterial to the flow field. The limits of integration in formulas corresponding to (3.7), (3.11), and (3.14) are now from $-h$ to $-H$. Furthermore we have

$$G_A(y/y') = \sum_2^\infty \frac{f_n f_n'}{q_n} + l + \sum_2^\infty \frac{\tanh Q_n l}{Q_n} F_n F_n', \qquad \text{(4.5) for (3.10)}$$

$$G_S(y/y') = \sum_2^\infty \left[\frac{f_n f_n'}{q_n} + \frac{\coth Q_n l}{Q_n} F_n F_n'\right], \qquad \text{(4.6) for (3.17)}$$

$$g_{rs}^A = \sum_{n=2}^\infty \frac{P_{nr} P_{ns}}{q_n} + l \delta_{1r} \delta_{1s} + \sum_{n=2}^\infty \frac{\tanh Q_n l}{Q_n} \delta_{nr} \delta_{ns}, \qquad \text{(4.7) for (3.15)}$$

$$g_{rs}^S = \sum_{n=2}^\infty \left[\frac{P_{nr} P_{ns}}{q_n} + \frac{\coth Q_n l}{Q_n} \delta_{nr} \delta_{ns}\right], \qquad \text{(4.8) for (3.18)}$$

where $\{f_n\}$ are given by (3.2) as before. Because of the changes in $\{F_n\}$ the modified forms of P_{nm} are:

$$P_{n1} = (2/D)^{\frac{1}{2}} (h - \sigma^{-1} \sin^2 q_n h)^{-\frac{1}{2}} \frac{\sin q_n D}{q_n} \tag{4.9a}$$

and

$$P_{nm} = 2(-1)^{m-1} [D(h - \sigma^{-1} \sin^2 q_n h)]^{-\frac{1}{2}} \frac{q_n \sin q_n D}{q_n^2 - Q_m^2}. \tag{4.9b}$$

5. Limiting cases

(a) Thin barrier, finite depth

For a barrier of zero thickness, fixed either on the bottom or in the free surface, the symmetric part, ϕ_S, corresponds to a vertical cliff extending for the entire depth of the fluid, and the reflexion coefficient, R_S, must be 1. This result also follows from the general formulas of the preceding sections. Taking the bottom obstacles, for instance, for small l, g^S_{11} behaves like l^{-1} while all other g^S_{rs} (r or $s \neq 1$) remain finite; thus $S_S \sim l^{-1}$ and $\lim_{l\to 0} R_S = 1$, as is evident from (3.18) and, with $(\,)_A$ changed to $(\,)_S$, (3.13) and (3.14). The limit of $l \to 0$ is easily taken for the antisymmetric part. The representation for $U_A(y)$ by (3.12) is still a useful one, although $\{F_i\}$ is now merely one of many orthonormal sets that may be adopted.

(*b*) *Infinite water depth, arbitrary thickness*

As $h \to \infty$, the eigenvalue spectrum in the zone $|x| > l$ for bottom obstacle contains a discrete value ($k = \sigma$, propagating) and a continuous part ($0 < q < \infty$, evanescent). The corresponding eigenfunctions are (Miles 1967)

$$f_1(y) = \surd(2\sigma)\, e^{\sigma y}, \tag{5.1}$$
$$f(y) = [2/\pi(q^2+\sigma^2)]^{-\frac{1}{2}}\,(q\cos qy + \sigma \sin qy),$$

which are orthonormal in the following sense:

$$\int_{-\infty}^{0} f_1^2\, dy = 1, \quad \int_{-\infty}^{0} f_1 f\, dy = 0, \quad \int_{-\infty}^{0} f(y,q) f(y,q')\, dy = \delta(q-q'). \tag{5.2}$$

The series involving f_n in (3.10) and (3.17) must be changed to integral form; hence

$$\begin{bmatrix} G_A \\ G_S \end{bmatrix} = \int_0^\infty \frac{dq}{q} ff' + \begin{bmatrix} \tan Kl \\ -\cot Kl \end{bmatrix} \frac{F_1 F_1'}{K} + \sum_{n=2}^{\infty} \begin{bmatrix} \tanh Q_n l \\ \coth Q_n l \end{bmatrix} \frac{F_n F_n'}{Q_n}. \tag{5.3}$$

The variational expressions for S_A or S_S are then obtained by substituting (5.3) into (3.11).

Although the general representation of U_A or U_S by (3.12) can be used, in view of Miles' (1967) success, in the case of a semi-infinite shelf, by using only the first term (the plane wave approximation), we record the corresponding approximation below:

$$\begin{bmatrix} S_A \\ S_S \end{bmatrix} = \frac{(K^2-\sigma^2)^2 e^{2\sigma H}}{4\sigma^3} \left\{ \frac{4}{\pi} \int_0^\infty dq \frac{q[q\sin qH + \sigma\cos qH]^2}{(q^2+\sigma^2)(q^2+K^2)^2} + \frac{1}{K} \begin{bmatrix} \tan Kl \\ -\cot Kl \end{bmatrix} (H + \sigma^{-1}\sinh^2 KH) \right\}. \tag{5.4}$$

The integral above has been evaluated in terms of tabulated functions by Miles (1967).

For surface obstacles, as $h \to \infty$, $D = h - H \to \infty$ also and the eigenvalue spectrum Q_n becomes continuous. The limiting expressions for G_A and G_S can be similarly written; however, a convenient and suitable trial function for $U_A(y)$ or $U_S(y)$ has not been found by the authors. The formulas will therefore be omitted.

6. Aspects of numerical computation

For all cases of finite water depth, the numerical work involves the summation of an infinite series for each g_{rs}, and the evaluation of determinants. From (3.3) and (3.14) it can be seen that, for large n, $q_n h \sim (n-1)\pi$ and $P_{nr} \sim q_n^{-1}$; hence

$P_{nr}P_{ns}/q_n \sim n^{-3}$ for given r and s, and the series for q_{rs} converges reasonably fast. However, owing to the factor $(q_n - Q_r)^{-1}$ in P_{nr}, a large (although finite) P_{nr} is possible. As an estimate we take n and r to be large, then $q_n \cong Q_r$ if

$$n \cong rh/H. \tag{6.1}$$

Therefore, the series cannot be truncated until beyond this threshold value. It also indicates that, as h/H increases, more terms must be summed to assure accuracy. Although these qualitative observations are made with respect to the bottom obstacles, the conclusions are true for surface obstacles as well.

Numerical tests were made to check the convergence of the results by varying the number of terms summed in each q_{rs} and the number of terms used to approximate $U(y)$ (equal to the order of determinants). Using 5 terms† for $U(y)$ and $5h/H + 15$ terms in the series (based on (6.1)) we obtain an accuracy within one per cent if $h/H \leqslant 10$ for bottom obstacles, and if $h/H \leqslant 6$ for surface obstacles. The slower convergence in the latter case may be expected because the leading terms in the assumed expansion for $U(y)$ become increasingly unsuitable for large h/H. All computations were done on an IBM 1130 computer.

For bottom obstacles and infinite h/H calculations were done only with a one-term (plane wave) approximation according to (5.4). Evidence of good accuracy is provided by comparing with Dean's (1945) exact solution for a thin barrier. Presumably, an even better approximation for Dean's case would be

$$U(y) \propto (H^2 - y^2)^{\frac{1}{2}},$$

which would give the scattering coefficients in terms of modified Bessel and Struve functions.

7. Results and discussions

(a) Bottom obstacles (figures 2, 3, 4, 5)

As in Newman (1965*b*), the prominent feature is the oscillatory nature of the reflexion coefficients resulting from the interaction between the two ends of the obstacle (figure 2). The oscillation increases with the obstacle length. We remark that the scattering of a semi-infinite shelf studied by Newman (1964*a*) and Miles (1967) cannot be obtained by taking the limit $l \to \infty$ from our treatment, since, from the standpoint of an initial-value problem, their problem corresponds to letting $l \to \infty$ before letting $t \to \infty$ whereas our formulation assumes the inverse. In figure 2 only the phase shift of the transmitted wave, θ_T, is plotted: that of the reflected wave θ_R can be obtained through the following relation:

$$\theta_R = \theta_T + \tfrac{1}{2}(-1)^N \pi, \quad \text{for} \quad k_N < k < k_{N+1}, \tag{7.1}$$

where k_N $(N = 1, 2, 3, \ldots)$ refer to the successive nodes of the reflexion coefficient (excluding $k = 0$). The sudden changes in θ_R can be explained by imagining a phasor diagram with the unit vectors $\mathbf{R}_S$ and $\mathbf{R}_A$ issuing from the origin. When the relative phase $\theta_A - \theta_S$ changes from less than π to greater than π,

† In similar electromagnetic wave-guide problems, a two-term approximation gives an accuracy within a few per cent (Collin 1960, p. 359).

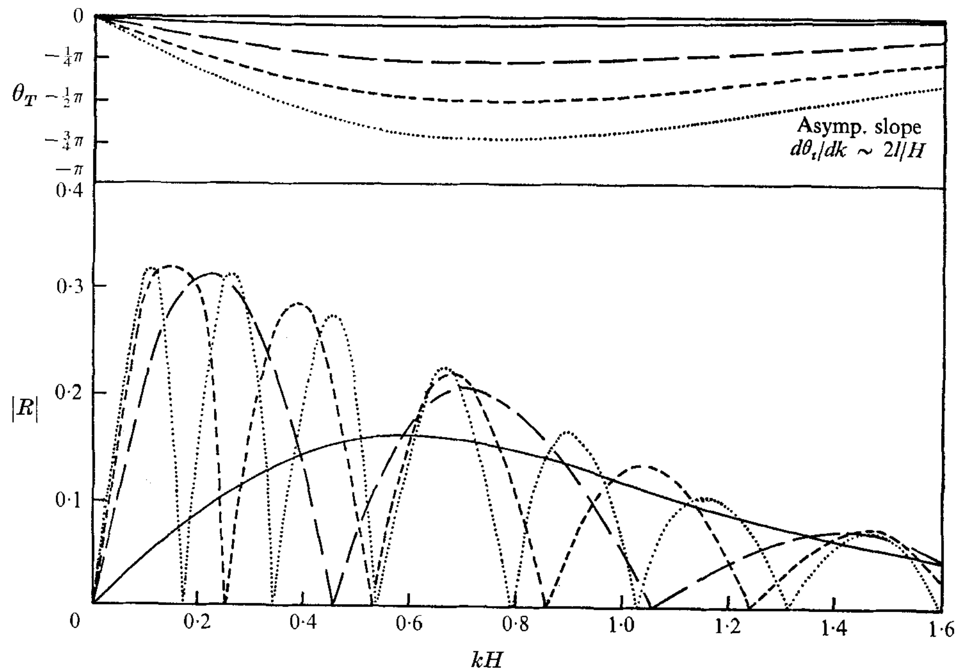

FIGURE 2. Reflexion coefficient and transmission phase angle for a submerged obstacle, $h/H = 2$: ——, $l/H = 0$; — — —, $l/H = 2$; – – –, $l/H = 4$; ······, $l/H = 6$.

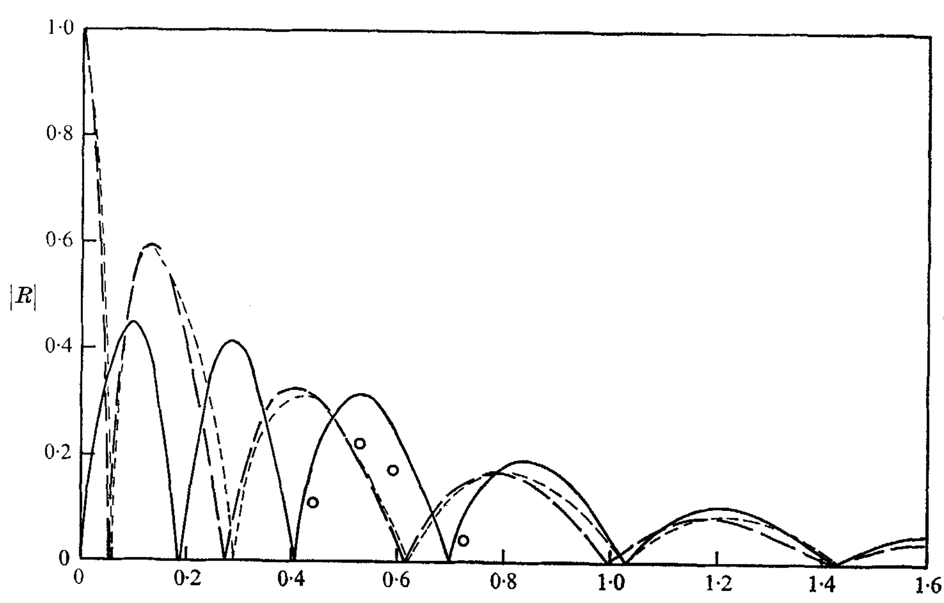

FIGURE 3. Reflexion coefficient for a submerged obstacle. For $l/H = 4{\cdot}43$, $h/H = 2{\cdot}78$: ——, theory; ○, Jolas' experiment. For $l/H = 4{\cdot}43$, $h/H = \infty$: — — —, theory; - - - -, Newman's approximation.

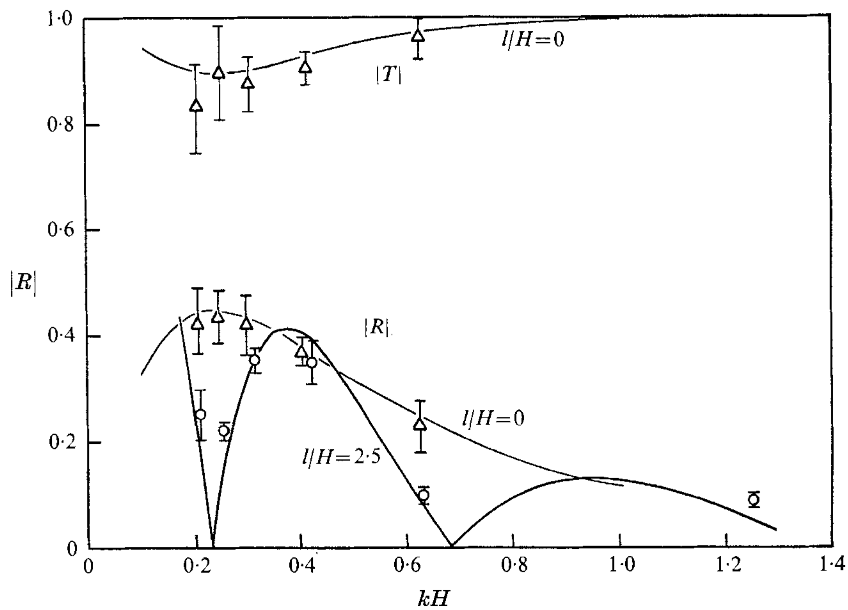

FIGURE 4. Reflexion and transmission coefficients for submerged obstacles. For $l/H = 2{\cdot}5$, $h/H = 5$: ○, Dick's experiment, ——, theory. For $l/H = 0$, $h/H = 5$: △, Dick's experiment, ——, theory.

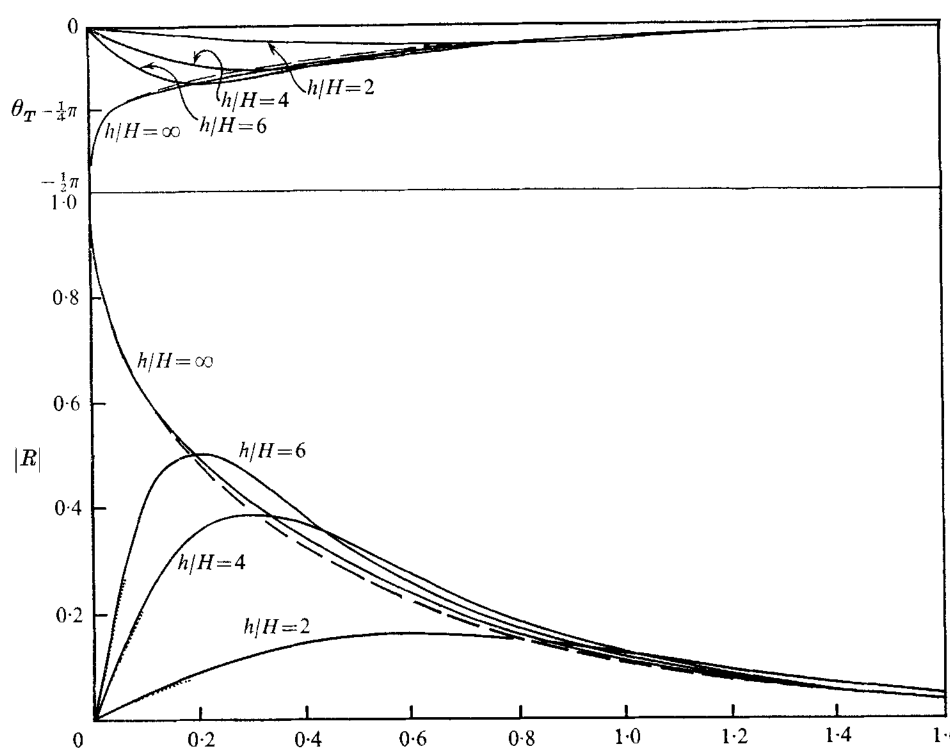

FIGURE 5. Reflexion coefficient and transmission phase angle for a submerged thin plate. — — —, Dean's exact solution for $h \to \infty$; ······, Ogilvie's long wave approximation.

$\mathbf{R} = \frac{1}{2}(\mathbf{R}_S + \mathbf{R}_A)$ changes its sense by π and hence $\theta_R - \theta_T$ varies from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ in principal value. Since $|R| = 0$ at $\theta_A - \theta_S = \pi$ these discontinuous changes occur at the nodes.

Comparison with Jolas' (1960) experiment and Newman's (1965*b*) corresponding approximate theory is made in figure 3. Newman's calculation was done on the basis of $h \to \infty$ while in the experiments h is finite. Hence two curves

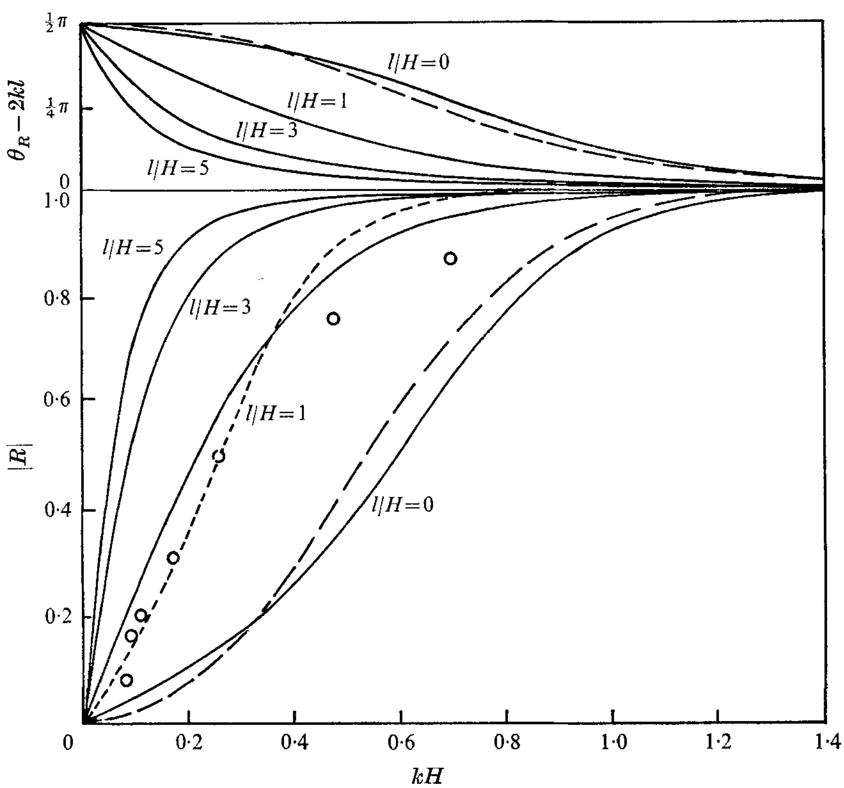

FIGURE 6. Reflexion coefficient and reflexion phase angle for a surface obstacle: ——— theory for $h/H = 2$, $l/H = 0, 1, 3, 5$; — — —, Ursell's exact solution for $h/H \to \infty$ $l/H = 0$; - - - -, theory for $h/H = 6$, $l/H = 1$. ○, Kincaid's (1960) experiment for $h/H = 6{\cdot}17$, $l/H = 1$.

are calculated from the present theory. For $h \to \infty$ our values are based on the plane wave approximation and the agreement with Newman is good for long and very short wavelengths, but rather crude for intermediate values. It should be remarked that, to use Newman's approach for finite h, the explicit knowledge of the semi-infinite shelf with finite h is needed, which is not yet available, see Miles (1967).

Recent experiments by Dick (1968) are plotted against our theory in figure 4. For the obstacle with finite length ($l/H = 2{\cdot}5$) higher harmonics are recorded on the transmission side and the experimental values of $|T|$ are not easy to define; direct comparison with the present theory is therefore not made here.

For submerged plate barriers (figure 5) it may be noted that the plane-wave approximation, (5.4), compares quite favourably with Dean's exact solution. Ogilvie's (1960) long wave approximation is also included and the agreement with the present theory is excellent.

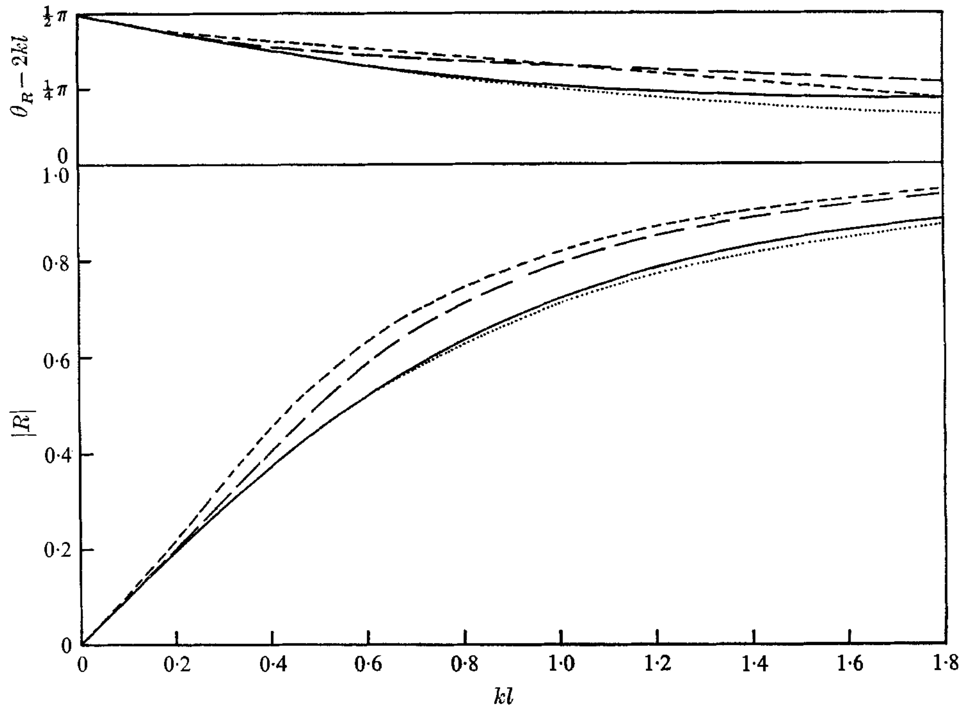

FIGURE 7. Reflexion coefficient and reflexion phase angle for a finite dock: ———, $h/l = 1$; — — —, $h/l = 3$; - - - -, $h/l = 5$; ······, Stoker's shallow water result.

(*b*) *Surface obstacles* (*figures* 6 *and* 7)

The effect of varying obstacle length is shown in figure 6 while the case of finite dock of zero draught is shown in figure 7. We find it convenient to plot the modified phase shift $\theta'_R \equiv \theta_R - 2kl$ instead of θ_R; physically the former amounts to measuring the phase angles with respect to the incoming edge $x = -l$. In all cases, $\theta_R = \theta_T + \frac{1}{2}\pi$. For a finite dock in a water of infinite depth Holford (1964) has obtained analytically $\theta'_R = \frac{1}{4}\pi + O(1/kl)$, which seems to differ from the standing-wave solution (hence $\theta'_R = 0$) of Friedrichs & Lewy (1949, equations (39), (40), p. 146) for a semi-infinite dock. Our calculated results indicate the tendency of $\theta'_R \to 0$ for large kl. Stoker's (1957, p. 434) formula based on linear long wave approximation is also plotted in figure 7. For $l/h = 1$ it can be seen that Stoker's result is surprisingly good for practically all values of kl.

In conclusion, the variational approach greatly facilitates the calculation of scattering properties of rectangular obstacles, of which only limiting cases have been treated heretofore.

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